Bayesian Statistics
Part III: Building Bayes Theorem
Part IV: Prior Specification

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Outline

Example

In 2012, the prevalence of HIV among the general U.S. population was estimated to be 0.38% (www.cdc.gov). Rapid diagnostic tests have been developed to test for the presence of HIV infection in as little as 10 minutes. Suppose one such diagnostic test was evaluated in a case-control study and it was observed that out of 1000 subjects with HIV, 967 obtained positive test results and out of 1000 subjects with out HIV, 985 obtained negative test results.

In summary

- $P(D+) = 0.0038 \Rightarrow P(D-) = 1 - 0.0038 = 0.9962$
- $P(T + | D+) = 0.967 \Rightarrow P(T - | D+) = 1 - 0.967 = 0.033$
- $P(T - | D-) = 0.985 \Rightarrow P(T + | D-) = 1 - 0.0985 = 0.015$

Suppose a person in the U.S. is selected at random, given the test, and the result is positive. Should this person begin treatment?

- $P(D + | T+)$?

Venn Diagram of the Example

Venn Diagram of the Example

Note

- $P(D + \cap T+) = P(D+)P(T + |D+)$
- $P(D - \cap T+) = P(D-)P(T + |D-)$
- $P(T+) = P(D+)P(T + |D+) + P(D-)P(T + |D-)$

$$P(D + |T+) = \frac{P(D+ \cap T+)}{P(T+)} = \frac{P(D+)P(T+|D+)}{P(D+)P(T+|D+)+P(D-)P(T+|D-)}.$$

Example (continued)

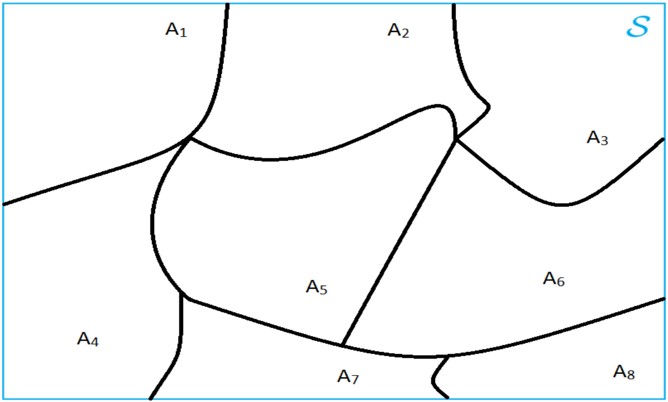
Recall

- $P(D+) = 0.0038 \Rightarrow P(D-) = 1 - 0.0038 = 0.9962$
- $P(T+|D+) = 0.967 \Rightarrow P(T-|D+) = 1 - 0.967 = 0.033$
- $P(T-|D-) = 0.985 \Rightarrow P(T+|D-) = 1 - 0.985 = 0.015$

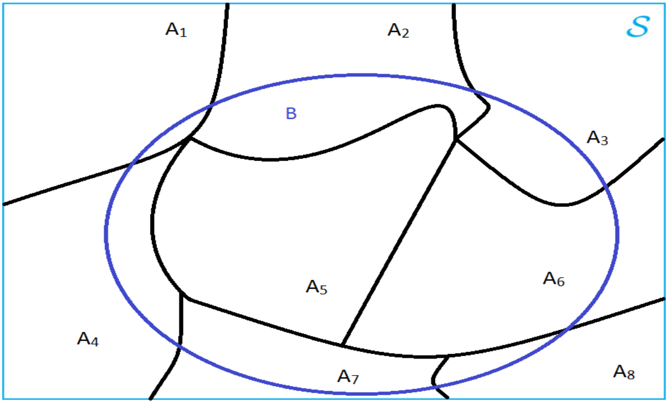
$P(D+|T+) = \frac{P(D+)P(T+|D+)}{P(D+)P(T+|D+)+P(D-)P(T+|D-)} =$
$\frac{(0.0038)(0.967)}{(0.0038)(0.967)+(0.9962)(0.015)} = 0.1974$

What does this mean?

More Generally...



More Generally...



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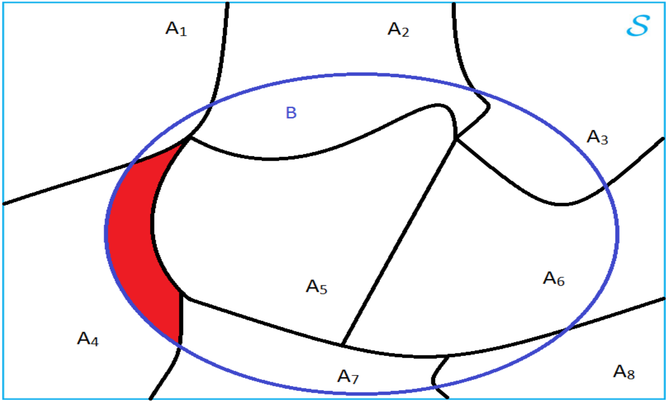

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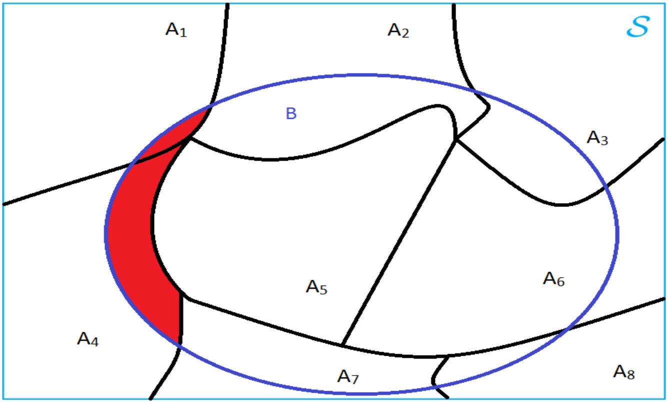

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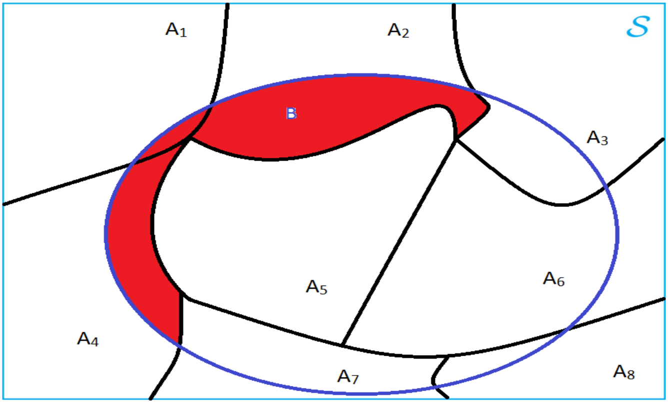

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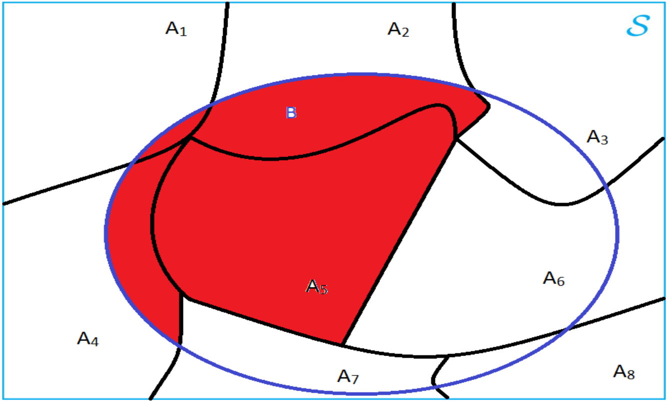

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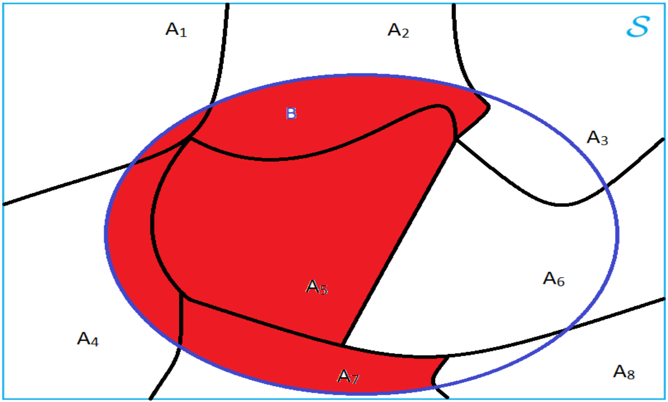

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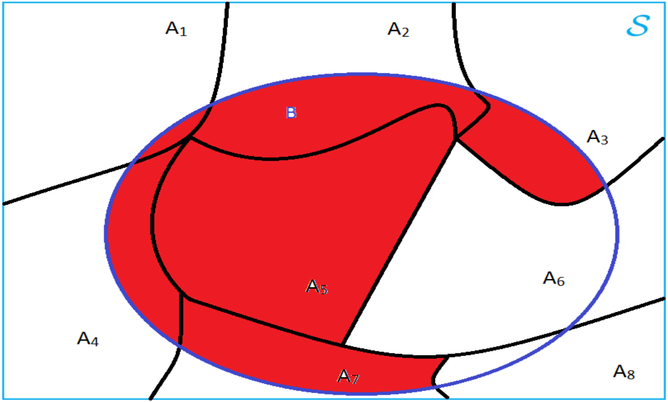

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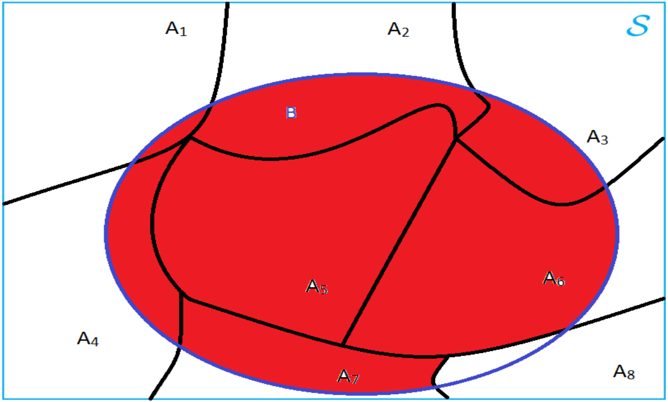
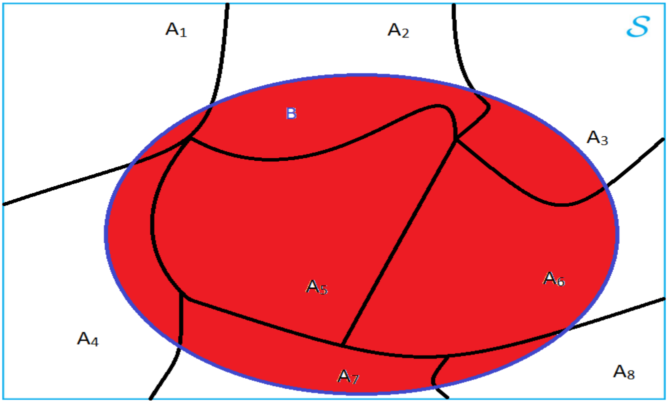


More Generally...



More Formally...

Bayes' Theorem

Let $A_1, \ldots, A_k$ be a partition of the sample space, and let B be any event in S. Then, for each $i = 1, \ldots, k$

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{k} P(A_j)P(B|A_j)} \tag{1}$$

- **Prior** probability of A_i.
- **Conditional** of B given A_i.
- **Total Probability**.
- **Posterior** probability of A_i.

Joint Probability Function

Suppose a *sample* is made up of independent observations $X_1, \ldots, X_n$ all assumed to belong to the same identical pdf (or pmf) $p(x|\theta)$. Then $X_1, \ldots, X_n$ are said to be i.i.d. (independent and identically distributed).

Joint pdf (or pmf)

The joint pdf (or pmf) of an i.i.d. *sample* $X = (X_1, \ldots, X_n)$ is given by

$$p(x_1, \ldots, x_n|\theta) = \prod p(x_i|\theta)$$

Likelihood

Likelihood Function

Let $p(x_1, \ldots, x_n|\theta)$ denote the joint pdf or pmf of the *sample* $X = (X_1, \ldots, X_n)$. Then, given $x_1, \ldots, x_n$ is observed, the function of θ defined by

$$L(\theta|x_1, \ldots, x_n) = p(x_1, \ldots, x_n|\theta)$$

is called the *likelihood function*

Plotting this function vs θ shows how plausible each θ value is. The maximum of the likelihood function is seen as the most plausible value of θ, given the data that was observed.

More Generally...

Bayes' Theorem

$$p(\theta|x_1,\dots,x_n) = \frac{p(\theta)p(x_1,\dots,x_n|\theta)}{\int p(\theta)p(x_1,\dots,x_n|\theta)d\theta} \quad (2)$$

- **Prior** distribution of θ.
- **Likelihood** of the data.
- Normalizing **constant**.
- **Posterior** distribution of θ.
- Not always tractable.
- Conjugate Priors.
- Gibbs Sampling and the Metropolis Algorithm.

Mean (Expected Value) of a Probability Distribution

- Discrete: $E(x) = \sum_x p(x)x$
  - Roll a single fair die infinitely many times. What is the mean of all the rolls?
  - $$E(x) = (1/6)1 + (1/6)2 + (1/6)3 + (1/6)4 + (1/6)5 + (1/6)6 = 3.5$$
- Continuous: $E(x) = \int_x p(x)xdx$
  - For a Normally distributed (continuous) outcome
  - $$E(x) = \int_x \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) xdx = \mu$$

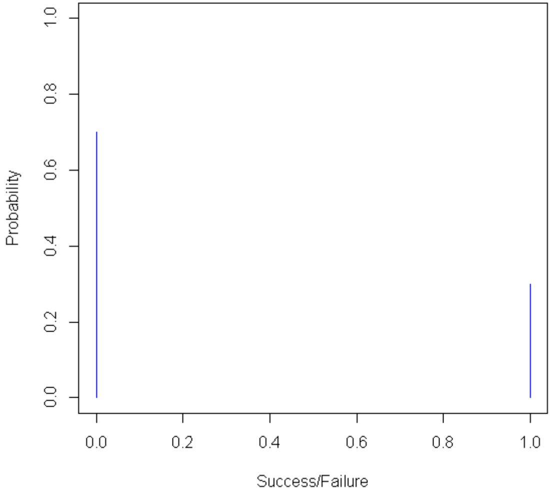
Variance of a Probability Distribution

- Discrete: $Var(x) = \sum_x p(x)(x - E(x))^2$
- Continuous: $Var(x) = \int_x p(x)(x - E(x))^2 dx$
- Note this is just the expected value of $(x - E(x))^2$.
- The positive square root of the variance is the standard deviation.
- **Major point:** The variance can represent our uncertainty about possible beliefs.
  - Less uncertainty about beliefs $\Rightarrow$ smaller variance.
  - Less certain about beliefs $\Rightarrow$ larger variance.

Bernoulli Distribution

- A Bernoulli experiment consists of a single trial with two possible outcomes (success/failure) with success probability π.
- $Bern(\pi)$
- $p(x) = \pi^x(1-\pi)^{1-x} \; x = 0, 1$
- $E(x) = \pi$
- $Var(x) = \pi(1-\pi)$
- Example: Plot a Bern(0.3)
  - `x<-c(0,1)`
  - `plot(x,dbinom(x,1,0.3),ylim=range(0,1),type="h", ylab="Probability",xlab="Success/Failure",col="blue")`

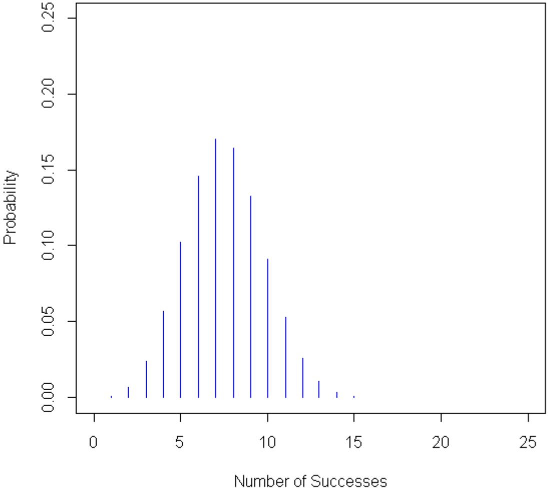
Bernoulli Distribution



Binomial Distribution

- A Bernoulli experiment repeated n times.
- Outcome of interest is the number of successes in those trials.
- $Bin(n, \pi)$
- $p(x) = \binom{n}{x} p^x (1-p)^{n-x} \; x = 0, 1, \ldots, n$
- $E(x) = n\pi$
- $Var(x) = n\pi(1-\pi)$
- Example: Plot a Bin(25,.3)
    - `x<-0:25`
    - `plot(x,dbinom(x,25,.3),ylim=range(0,.25),type="h", ylab="Probability",xlab="Number of Successes",col="blue")`

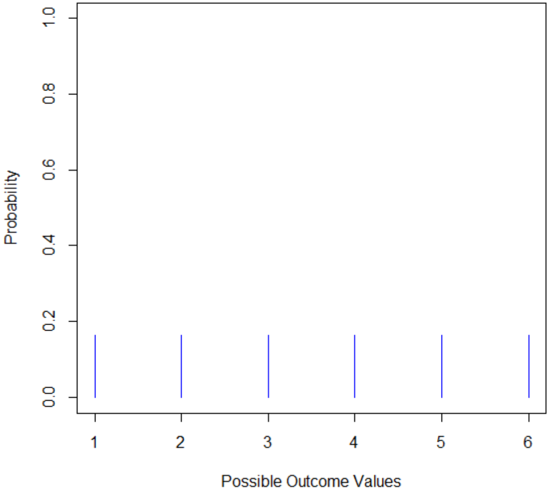
Binomial Distribution



Discrete Uniform Distribution

- Puts equal probability on every observable outcome.
- $DUniform(N)$
- $p(x) = \frac{1}{N}\ x = 1, 2, \ldots, N$
- $E(x) = \frac{N+1}{2}$
- $Var(x) = \frac{(N+1)(N-1)}{12}$
- Example: Plot a discrete uniform for the roll of a fair die.
  - `x<-1:6`
  - ```
    plot(x,rep(1/length(x),length(x)),ylim=range(0,1),type="h",
    ylab="Probability",xlab="Possible Outcome
    Values",col="blue")
    ```
    ```

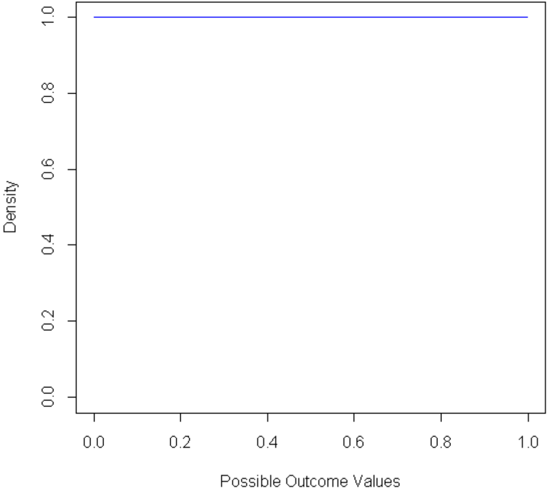
Discrete Uniform Distribution



Uniform Distribution

- Puts equal density on every subinterval of the same length between two points $[a, b]$.
- $Uniform(a, b)$
- $p(x) = \frac{1}{b-a}\ a \leq x \leq b$
- $E(x) = \frac{b+a}{2}$
- $Var(x) = \frac{(b-a)^2}{12}$
- Example: Plot a Uniform(0,1)
  - `x<-seq(0,1,length=1000)`
  - `plot(x,dunif(x,0,1),ylim=range(0,1),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

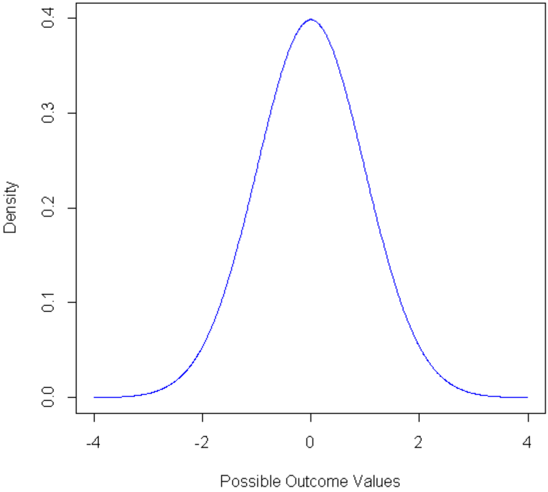
Uniform Distribution



Normal Distribution

- Symmetric, bell-shaped curve.
- Outcome of interest could have any value in the real numbers.
- $N(\mu, \sigma^2)$
- $p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \ -\infty < x < \infty$
- $E(x) = \mu$
- $Var(x) = \sigma^2$
- Example: Plot a N(0,1)
  - `x<-seq(-4,4,length=1000)`
  - `plot(x,dnorm(x,0,1),ylim=range(0,.25),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

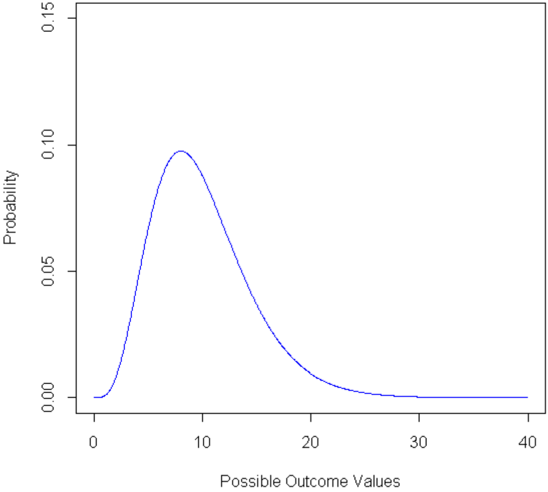
Normal Distribution



Gamma Distribution

- Sometimes used to model lifetimes. Usually right-skewed. Often used as a prior for $1/Var(x)$.
- Outcome of interest must be non negative.
- $Gamma(\alpha, \beta)$
- $p(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$ $0 < x < \infty$
- $E(x) = \alpha\beta$
- $Var(x) = \alpha\beta^2$
- Example: Plot a Gamma(5,2)
  - `x<-seq(0,40,length=1000)`
  - `plot(x,dgamma(x,5,.5),ylim=range(0,.15),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

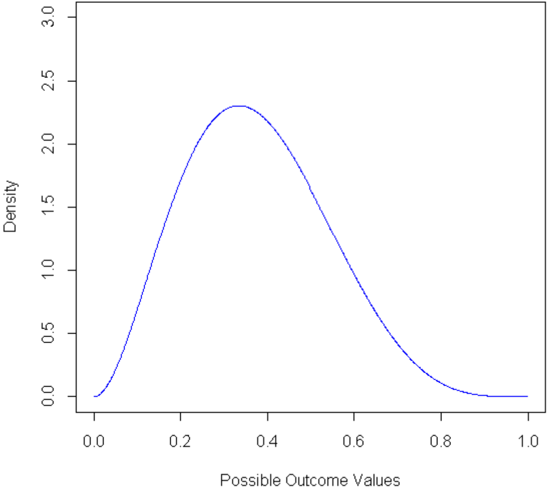
Gamma Distribution



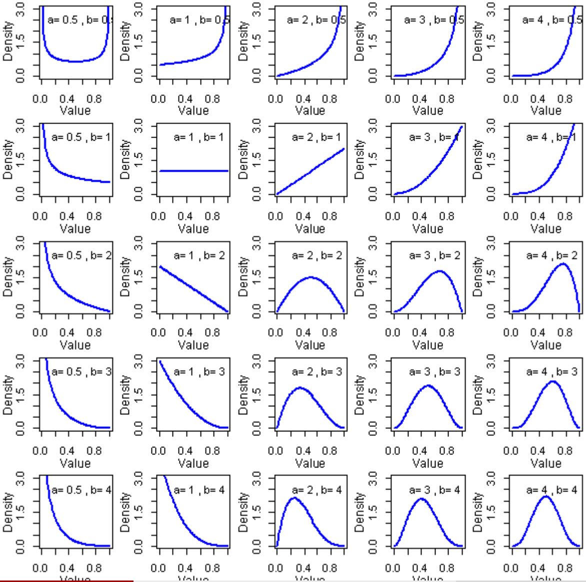
Beta Distribution

- Often used to model probabilities or prevalences.
  Often used as a prior for these quantities.
- Outcome of interest lives on the interval $[0, 1]$.
- $Beta(\alpha, \beta)$
- $p(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \; 0 < x < \infty$
- $E(x) = \frac{\alpha}{\alpha+\beta}$
- $Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$
- Example: Plot a Beta(3,5)
  - `x<-seq(0,1,length=1000)`
  - `plot(x,dbeta(x,3,5),ylim=range(0,3),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

Beta Distribution



Beta Distribution



Conjugate Priors

Posterior has the same distributional form as the prior, then the prior is conjugate for the likelihood.

Table: Common Conjugate Priors and Corresponding Likelihoods.

Likelihood	Prior	Posterior
Bernoulli	Beta	Beta
Binomial	Beta	Beta
Normal	Normal	Normal
Poisson	Gamma	Gamma

Lung Cancer Data Revisited

Table: Gender of Subjects with Lung Cancer; 1=Female

Gender	Gender
0	0
1	0
0	0
0	1
1	1
0	0
0	0
0	1
0	0
0	1
1	0
0	0

- Note that $x_i \in \{0, 1\}$
- Each observation $x_i \sim Bern(\theta)$
- $p(x_i|\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$
- $L(\theta|x_i) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$
- To ease notation let $y = \sum x_i$

- Prior: $p(\theta) = Beta(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$
- Mean: $\frac{\alpha}{\alpha+\beta}$
- Variance: $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Lung Cancer Data Revisited

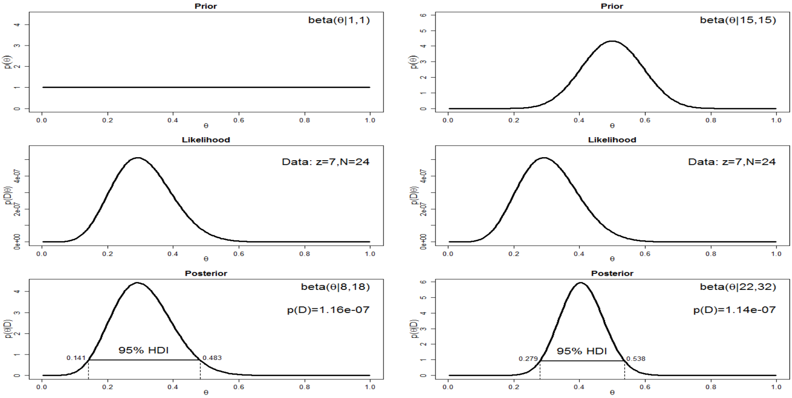
- Posterior: $P(\theta|y) = \frac{p(\theta)p(y|\theta)}{\int p(\theta)p(y|\theta)d\theta} = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{y}(1-\theta)^{n-y}}{\int \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{y}(1-\theta)^{n-y}d\theta}$
$= \frac{\theta^{\alpha-1+y}(1-\theta)^{\beta-1+n-y}}{\int \theta^{\alpha-1+y}(1-\theta)^{\beta-1+n-y}d\theta} = \frac{\Gamma(y+\alpha+n-y+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}$

$= Beta(\alpha + y, n - y + \beta)$

- Mean: $\frac{\alpha+y}{\alpha+\beta+n} = \left(\frac{\alpha+\beta}{\alpha+\beta+n}\right)\left(\frac{\alpha}{\alpha+\beta}\right) + \left(1 - \frac{\alpha+\beta}{\alpha+\beta+n}\right)\left(\frac{y}{n}\right)$
- Variance: $\frac{(\alpha+y)(n-y+\beta)}{(\alpha+\beta+n)^2(\alpha+\beta+2n)}$

Lung Cancer Data Revisited

Consider a $Beta(1,1)$ and $Beta(15,15)$ as possible priors



- Mean : 0.308 0.407
- Var : 0.008 0.004

Prior Beliefs

See file “Prior beliefs.pptx”